CHAPTER 8
A Description of Projective Contractions in the Orlicz-Kantorovich Lattice

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Abstract. In the present paper we show that any positive projective contractions $Q$ with $Q1 = 1$ in the Orlicz—Kantorovich lattices $L_M(\mathcal{P}, \mu)$ can be represented in the form $Q(f)(\omega) = E_\omega(f(\omega) | \mathcal{P}^1_{\omega})$ for any $f \in L_M(\mathcal{P}, \mu)$ and for almost all $\omega \in \Omega$, where $E_\omega(\cdot | \mathcal{P}^1_{\omega})$ is conditional expectation operator. Using this result we get abstract characterization conditional expectation operators in the Orlicz-Kantorovich $L_M(\mathcal{P}, \mu)$—lattice.

Keywords: Orlicz-Kantorovich lattice; positive projective contraction; conditionally expectation operator.
1 Introduction

One of the important problems of positive operator’s theory is an abstract characterization of the conditional expectation operators in function spaces. In (Rao, 1976) a characterization of the conditional probability measures as subclasses of vector measures on general Banach function spaces is given. Moreover the following result is proven;

**Theorem 1.1.** (Rao, 1976). Let \((\Omega, \Sigma, \mu)\) be a measurable space with a finite measure \(\mu\). If \(T: L_p(\mu) \to L_p(\mu), (1 \leq p < \infty)\) is a positive projective contraction with \(T1 = 1\), then \(Tf = E(f \mid F), f \in L_p(\mu)\), for a unique \(\sigma\)-subalgebra \(F \subset \Sigma\). Where \(E(\cdot \mid F)\) is conditional expectation operator relative to \(F\).

In (Rao, 1965) this theorem is proven for Orlicz spaces. In (Rao, 1976), necessary and sufficient conditions for \(T: L_1(\mu) \to L_1(\mu)\) to be conditional expectation operator relative to \(F\) is obtained. Dodds, Huijsmans and De Pagter in (Dodds et al., 1965) extended these result to the vector lattices. We recall that in the theory of Banach bundles \(L_0\)-valued Banach spaces are considered, and such spaces are called Banach–Kantorovich spaces. In (Gutman, 1993), (Gutman, 1995) the theory of Banach–Kantorovich spaces is developed. Analogues of many well-known functional spaces have been defined and studied. For example, in (Ganiev, 2006) Banach-Kantorovich lattice \(L_p(\mathcal{P}, \bar{\mu})\) represented as a measurable bundle of classical \(L_p\)–lattices. In (Zakirov & Chilin, 2009), (Zakirov, 2007) an analogue of the Orlicz spaces has been considered. Naturally, these functional Kantorovich spaces should have many properties similar to the classical ones constructed by the real valued measures.

To investigate the properties of Banach–Kantorovich spaces it is natural to use measurable bundles of such spaces. Since, one has a sufficiently well explored theory of measurable bundles of Banach lattices (Ganiev, 2006), it is an effective tool which gives well opportunity to obtain various properties of Banach–Kantorovich spaces. It is worthy to mention that using this way, weighted ergodic theorems for positive contractions of Banach-Kantorovich lattices \(L_p(\mathcal{P}, \bar{\mu})\) have been established (Chilin & Ganiev (2000)), ( Ganiev & Mukhamedov, 2013).

**Definition 1.2.** The \(L_0\)–linear, \(L_0\)–bounded positive operator \(T\) from \(L_p(\mathcal{P}, \bar{\mu})\) onto \((ba)\)–closest vector subspace \(L_p(\mathcal{P}^1, \bar{\mu}^1)\) of \(L_p(\mathcal{P}, \bar{\mu})\) is said to be
conditional expectation operator with respect to the regular Boolean sub-algebra $\mathcal{V}^1$ if $\int T(\hat{f}) \, d\mu = \int \hat{f} \, d\mu$ and it is denoted by $T = E(\cdot | \mathcal{V}^1)$.

In (Kusraev, 1985), Theorem 4.2.9 it has been proven that there exists conditional expectation operator $E(\cdot | \mathcal{V}^1) : L_1(\mathcal{P}, \hat{\mu}) \rightarrow \mathcal{A}(\mathcal{V}^1, \hat{\mu})$ satisfying the following conditions:

1) $E(\cdot | \mathcal{V}^1)$ is linear, positive, idempotent operator;

2) $\int E(\hat{f} | \mathcal{V}^1) \, d\mu = \int \hat{f} \, d\mu$;

3) $E(\hat{g} \hat{f} | \mathcal{V}^1) = \hat{g} E(\hat{f} | \mathcal{V}^1)$ for any $\hat{g} \in L^\infty(\mathcal{V}^1, \hat{\mu})$ and $\hat{f} \in L_1(\mathcal{P}, \hat{\mu})$ for any $\hat{f} \in L_1(\mathcal{P}, \hat{\mu})$ and $E(1) = 1$.

It means that $E(\cdot | \mathcal{V}^1)$ is projective contraction in the Banach — Kantorovich lattice $L_1(\mathcal{P}, \hat{\mu})$. In this case $\| E(\hat{f} | \mathcal{V}^1) \|_{L_1(\mathcal{P}, \hat{\mu})} \leq \| \hat{f} \|_{L_1(\mathcal{P}, \hat{\mu})}$ for any $\hat{f} \in L_1(\mathcal{P}, \hat{\mu})$.

Let Banach-Kantorovich lattice $L_p(\mathcal{P}, \hat{\mu})$ be represented as a measurable bundle of classical $L_p(\mathcal{V}_\omega, \mu_\omega)$—lattices. The description of conditional expectation operator $E(\cdot | \mathcal{V}^1) : L_1(\mathcal{P}, \hat{\mu}) \rightarrow \mathcal{A}(\mathcal{V}^1, \hat{\mu})$ is obtained in (Ganiev, 2006).

**Theorem 1.3.** Let $E(\cdot | \mathcal{V}^1) : L_1(\mathcal{P}, \hat{\mu}) \rightarrow \mathcal{A}(\mathcal{V}^1, \hat{\mu})$ be conditional expectation operator. Then for any $\omega \in \Omega$ there exists $E\omega(\cdot | \mathcal{V}_\omega^1) : L_1(\mathcal{V}_\omega, \mu_\omega) \rightarrow L_1(\mathcal{V}_\omega, \mu_\omega)$ conditionally expectation operator, such that $E(\hat{f} | \mathcal{V}^1)(\omega) = E\omega(\hat{f}(\omega) | \mathcal{V}_\omega^1)$ for any $\hat{f} \in L_1(\mathcal{P}, \hat{\mu})$ and for almost all $\omega \in \Omega$, where $E\omega(\cdot | \mathcal{V}_\omega^1)$ is conditional expectation operator on $L_p(\mathcal{V}_\omega, \mu_\omega)$.

Consequences of the development of the general theory, conditional expectation operators in Banach — Kantorovich lattices $L_p(\mathcal{P}, \hat{\mu})$ over the ring of measurable functions gives rise the problem of an abstract characterization conditional expectation operators in Banach — Kantorovich lattices $L_p(\mathcal{P}, \hat{\mu})$ which are reasonably solved using the method of measurable bundles. In the present paper we will show that any positive projective contractions $Q$ with $Q1 = 1$ in the Orlicz — Kantorovich lattices $L_M(\mathcal{P}, \hat{\mu})$ can be represented in the form $Q(\hat{f})(\omega) = E\omega(\hat{f}(\omega) | \mathcal{V}_\omega^1)$ for any $\hat{f} \in L_M(\mathcal{P}, \hat{\mu})$ and for almost all $\omega \in \Omega$, where $E\omega(\cdot | \mathcal{V}_\omega^1)$ is conditional expectation operator. To prove the main result of this paper we are going to use measurable bundles of Banach — Kantorovich lattices. We note that one of the effective methods to study of Banach — Kantorovich spaces is measurable bundles (Gutman, 1995).
In (Ganiev & Mukhamedov, 2013) prove weighted ergodic theorems and multiparameter weighted ergodic theorems for positive contractions acting on $L_p(\mathcal{P}, \mu)$. In (Ganiev & Mukhamedov, 2015) this results generalized for Orlicz-Kantorovich $L_m(\mathcal{P}, \mu)$-lattice.

2 Preliminaries

In this section we recall necessary definitions and results concerning Banach-Kantorovich lattices.

Let $(\Omega, \Sigma, \mu)$ be a space with complete finite measure, $L_0 = L_0(\Omega)$ be the algebra of classes of measurable functions on $(\Omega, \Sigma, \mu)$. Consider a real vector space $E$.

A transformation $\| \cdot \| : E \rightarrow L_0$ is called vector-valued or $L_0$-valued norm on $E$, if it satisfies the following conditions:

i) $\| x \| \geq 0$ for all $x \in E$; $\| x \| = 0 \iff x = 0$;

ii) $\| \lambda x \| = |\lambda| \| x \|$ for all $\lambda \in \mathbb{R}$, $x \in E$;

iii) $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in E$.

A pair $(E, \| \cdot \|)$ is said to be a lattice-normed space (LNS) over $L_0$.

An LNS $E$ is disjunctively decomposed or shortly, $d$ — decomposed, if the following axiom is fulfilled:

For any $x \in E$ and disjunct elements $e_1, e_2 \in L_0$, satisfying $\| x \| = e_1 + e_2$, there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$, $\| x_1 \| = e_1$ and $\| x_2 \| = e_2$.

A net $\{x_\alpha\} \in E$ is $(bo)$—convergent to $x \in E$, if a net $\{\| x_\alpha - x \|\}$ is $(o)$ — convergent to $L_0$.

We say that an LNS is $(bo)$—complete, if any $(bo)$ — fundamental net $\{x_\alpha\}$ $(bo)$ — converges to some element of this space.

Any $d$—decomposable and $(bo)$ — complete LNS over $L_0$ is said to be a Banach-Kantorovich space (BKS) over $L_0$ (Kusraev, 1985).
If a Banach-Kantorovich space is simultaneously a vector lattice and the norm is monotone, then it becomes a Banach — Kantorovich lattice.

Let $X$ be a mapping, which maps every point $\omega \in \Omega$ to some Banach space $(X(\omega), \| \cdot \|_{X(\omega)})$. In what follows, we assume that $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A function $u$ is said to be a section of $X$, if it is defined almost everywhere in $\Omega$ and takes its value $u(\omega) \in X(\omega)$ for $\omega \in \text{dom}(u)$, where $\omega \in \text{dom}(u)$ is the domain of $u$.

Let $L$ be some set of sections.

**Definition 2.1.** (Gutman, 1995). A pair $(X, L)$ is said to be a measurable bundle of Banach spaces over $\Omega$ if

i. $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2 : \omega \in \text{dom}(c_1) \cap \text{dom}(c_2) \rightarrow \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;

ii. The function $||c|| : \omega \in \text{dom}(c) \rightarrow ||c(\omega)||_{X(\omega)}$ is measurable for all $c \in L$;

iii. For every $\omega \in \Omega$ the set $\{c(\omega) : c \in L, \omega \in \text{dom}(c)\}$ is dense in $X(\omega)$;

A measurable Banach bundle $(X, L)$ is called measurable bundle of Banach lattices (MBBL), if $(X(\omega), \| \cdot \|_{X(\omega)})$ are Banach lattices for all $\omega \in \Omega$ and all $c_1, c_2 \in L c_1 \vee c_2 \in L$, where $c_1 \vee c_2 : \omega \in \text{dom}(c_1) \cap \text{dom}(c_2) \rightarrow c_1(\omega) \vee c_2(\omega)$.

Henceforth, $(X, L)$ will be denoted just by $X$.

A section $s$ is a step-section, if there are pairwise disjoint sets $A_1, A_2, ..., A_n \in \Sigma$ and sections $c_1, c_2, ..., c_n \in L$ such that $\bigcup_{i=1}^n A_i = \Omega$ $\Rightarrow$ $s(\omega) = \sum_{i=1}^n X_{A_i}(\omega) c_i(\omega)$ for almost all $\omega \in \Omega$.

A section $u$ is measurable, if for any $A \in \Sigma$ there is a sequence $s_n$ of step-sections such that $s_n(\omega) \rightarrow u(\omega)$ for almost all $\omega \in A$.

Let $M(\Omega, X)$ be the set of all measurable sections. By symbol $L_0(\Omega, X)$ we denote factorization of the $M(\Omega, X)$ with respect to almost everywhere equality. Usually, by $\bar{u}$ we denote a class from $L_0(\Omega, X)$, containing the
section $\mu \in M(\Omega, X)$, and by $\|\hat{u}\|$ we denote the element of $L_0(\Omega)$, containing $\|u(\omega)\|_{X(\omega)}$.

Let $X$ be an MBBL. We set $\hat{u} \leq \hat{v}$, if $u(\omega) \leq v(\omega)$ a.e. One can easily show that the relation $\hat{u} \leq \hat{v}$ constitutes a partial order on $L_0(\Omega, X)$.

If $X$ is an MBBL, then $L_0(\Omega, X)$ is a Banach-Kantorovich lattice (Chilin & Ganiev, 2000).

Let $V_\omega, \omega \in \Omega$ be a family of complete Boolean algebras with strictly positive real-valued measures $\mu_\omega$. We set $\rho_\omega(e, g) = \mu_\omega(e \wedge g), e, g \in V_\omega$. Then $(V_\omega, \mu_\omega)$ are complete metric spaces. Consider the transformation $V$, which assigns some Boolean algebra $V_\omega$ to every point $\omega \in \Omega$. Let $L$ be a non-empty set of sections $V$.

**Definition 2.2.** A pair $(V, L)$ is called a measurable bundle of boolean algebras over $\Omega$, if

i) $(V, L)$ is a measurable bundle of metric spaces (Chilin & Ganiev, 2000);

ii) If $e \in L$, then $e^+ \in L$, where $e^+ : \omega \in \text{dom}(e) \rightarrow e^+(\omega)$;

iii) If $e_1, e_2 \in L$, then $e_1 \vee e_2 \in L$, where

$$e_1 \vee e_2 : \omega \in \text{dom}(e_1) \cap \text{dom}(e_2) \rightarrow e_1(\omega) \vee e_2(\omega)$$  \hspace{1cm} (1)

Let $M(\Omega, V)$ be the set of measurable sections, $V$-factorization of $M(\Omega, V)$ with respect to almost everywhere equality. Define a transformation $\hat{\mu} : V \rightarrow L_0(\Omega)$ by $\hat{\mu}(e) = \mu_\omega(\text{int}(f) = \mu_\omega(\text{int}(f) = \mu_\omega(e(\omega)))$. Evidently, $\hat{\mu}$ is well defined. It is well known that $(V, \hat{\mu})$ is a complete boolean algebra with strictly positive $L_0(\Omega)-$valued modulated measure $\hat{\mu}$, moreover, the boolean algebra $V(\Omega)$ of all idempotents from $L_0(\Omega)$ is identified with regular sub-algebra in $V$ and $\hat{\mu}(g \hat{e}) = g\mu_\omega(\text{int}(f) = \mu_\omega(\text{int}(f) = \mu_\omega(e(\omega)))$ for all $g \in V(\Omega)$ and $\hat{e} \in V$. By $L_0(V, \hat{\mu})$ we denote an order complete vector lattice $C_\omega(Q(V))$, where $Q(V)$ is the Stonian compact associated with complete Boolean algebra $V$. Following the well-known scheme of the construction of $L_0$-spaces, a space $L_p(V, \hat{\mu})$ can be defined by

$$L_p(V, \hat{\mu}) = \{ f \in L_0(V, \hat{\mu}) \mid \int |f|^p d\hat{\mu} \text{ exist} \}, \ p \geq 1$$  \hspace{1cm} (2)

where $\hat{\mu}$ is an $L_0(\Omega)$-valued measure on $V$. 

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It is known (Kusraev, 1985) that \( L_p(\tilde{\mathcal{V}}, \tilde{\mu}) \) is a BKS over \( L_0(\Omega) \) with respect to the \( L_0(\Omega) \)-valued norm \( \| \tilde{f} \|_{L_p(\mathcal{V}, \mu)} = (\int |\tilde{f}|^p d\tilde{\mu})^{1/p} \). Moreover, \( L_p(\tilde{\mathcal{V}}, \tilde{\mu}) \) is a Banach-Kantorovich lattice (Kusraev, 1985).

An even continuous convex function \( \mathcal{V} \mapsto \tilde{\mathcal{V}}(\mathcal{V}) \) is called an \( N \)-function, if
\[
\mathcal{V}(\mathcal{V}) = \lambda \quad \text{and} \quad \mathcal{V}(\mathcal{V}) = \mathcal{V}(\mathcal{V}) \text{ for all } \mathcal{V} \in L_0^N.
\]
The set \( L_0^N(\mathcal{V}, \tilde{\mu}) := \{ x \in \mathcal{V}(\mathcal{V}): M(x) \in L_1(\mathcal{V}, \tilde{\mu}) \} \) (3)
is called the Orlicz \( L_0 \)-class, and the vector space
\[
L_M := L_M(\mathcal{V}, \tilde{\mu}) := \{ x \in \mathcal{V}(\mathcal{V}): x_1 \in \mathcal{V}(\mathcal{V}) \text{ for all } y \in L_0^N \} \quad \text{(4)}
is called the Orlicz \( L_0 \)-space, where \( N \) is the complementary \( N \)-function to \( M \).

We notice that, \( L_M(\mathcal{V}, \tilde{\mu}) \subset \mathcal{V}(\mathcal{V}) \).

Define the \( L_0 \)-valued Orlicz norm on \( L_M(\mathcal{V}, \tilde{\mu}) \) as follows:
\[
\| x \|_M := \sup \{ \| \mathcal{V}(x) y d\tilde{\mu} \|: y \in A(N) \}, \quad x \in L_M(\mathcal{V}, \tilde{\mu}),
\]
where \( A(N) = \{ y \in L_0^N: \int N(y) d\tilde{\mu} \leq 1 \} \) and \( 1 \) is identity element of \( L_0 \). The pair \( (L_M(\mathcal{V}, \tilde{\mu}), \| \|_M) \) is a Banach–Kantorovich lattice which is called the Orlicz–Kantorovich lattice associated with the \( L_0 \)-valued measure (Zakirov & Chilin, 2009), (Zakirov, 2007).

**Theorem 2.3.** (Zakirov & Chilin, 2009). If the \( N \)-function \( M \) meets the \( \Delta_2 \)-condition then the Orlicz–Kantorovich lattice \( L_M(\mathcal{V}, \tilde{\mu}) \) is isometrically and order isomorphic to \( L_0(\Omega, X) \), where \( (X, L) \) is the measurable Banach bundle over \( \Omega \) such that \( X(\omega) = L_M(\mathcal{V}_\omega, \mu_\omega) \) and
\[
L = \{ \sum_{i=1}^n \alpha_i e_i: \alpha_i \in \mathbb{R}, e_i \in M(\Omega, \mathcal{V}), i = 1, \ldots, n, n \in \mathbb{N} \}.
\]
3 A Description of Projective Contractions in the Orlicz-Kantorovich lattice

In this section we will show that any positive projective contractions $Q$ with $Q1 = 1$ in the Orlicz—Kantorovich lattices $L_M(\mathcal{V}, \mu)$ can be represented in the form

$$Q(\hat{f})(\omega) = E_\omega(f(\omega)|\mathcal{V}_\omega^1)$$

(7)

for any $\hat{f} \in L_M(\mathcal{V}, \mu)$ and for almost all $\omega \in \Omega$, where $E_\omega(\cdot|\mathcal{V}_\omega^1)$ is conditional expectation operator.

Proposition 3.1. Let $M$ be an $N$-function, and $E(\cdot|\mathcal{V}^1): L_1(\mathcal{V}, \mu) \to L_1(\mathcal{V}^1, \mu^1)$ be conditionally expectation operator. Then

$$E(L_M(\mathcal{V}, \mu) \mathcal{V}^1) \subset L_M(\mathcal{V}, \mu)$$

(8)

and

$$\|E(\cdot|\mathcal{V}^1)\|_{L_M(\mathcal{V}, \mu)} \leq \|\hat{f}\|_{L_1(\mathcal{V}, \mu)}$$

for any $\hat{f} \in L_1(\mathcal{V}, \mu)$ and $E(1|\mathcal{V}^1) = 1$ by Proposition 3.1 (Zakirov & Chilin, 2009)

$$E(L_M(\mathcal{V}, \mu) \mathcal{V}^1) \subset L_M(\mathcal{V}, \mu)$$

(10)

As

$$\|E(\hat{f}|\mathcal{V}^1)\|_{M}(\omega) = \|E(\hat{f}|\mathcal{V}^1)(\omega)\|_{L_M(\mathcal{V}_\omega, \mu|\omega)} = \|E_\omega(f(\omega)|\mathcal{V}_\omega^1)\|_{L_M(\mathcal{V}_\omega, \mu|\omega)} \leq \|f(\omega)\|_{L_M(\mathcal{V}_\omega, \mu|\omega)} = \|\hat{f}\|_{M}(\omega)$$

(11)

a.e. we get
\[\| E(\hat{f} | \mathcal{P}^1) \|_M \leq \| \hat{f} \|_M \]  \hspace{1cm} (12)

or

\[\| E(\cdot | \mathcal{P}^1) \|_{L_M(\mathcal{P}, \mathcal{P})} \leq 1.\]  \hspace{1cm} (13)

As

\[\| E_\omega(f(\omega)|\mathcal{P}_\omega^1) \|_{L_M(\mathcal{P}_\omega, \mathcal{P}_\omega)} = \| f(\omega) \|_{L_M(\mathcal{P}_\omega, \mathcal{P}_\omega)}\]

for almost all \( \omega \in \Omega \) and

for any \( \{f(\omega)\}_{\omega \in \Omega} = \hat{f} \) with \( f(\omega) \in L_M(\mathcal{P}_\omega^1, \mathcal{P}_\omega^1) \) we have that

\[\| E(\cdot | \mathcal{P}^1) \|_{L_M(\mathcal{P}, \mathcal{P})} = 1.\]  \hspace{1cm} (14)

Let N-function \( M \) is said to satisfy \( \Delta_2^{-}\)-condition.

**Theorem 3.2.** Let \( Q : L_M(\mathcal{P}, \mathcal{P}) \to L_M(\mathcal{P}, \mathcal{P}) \) be a linear positive operator. If

1. \( Q^2 = Q; \)

2. \( \| Q \|_{L_1(\mathcal{P}, \mathcal{P})} \leq 1; \)

3. \( Q(1) = 1; \)

then

i. \( \| Q \|_{L_M(\mathcal{P}, \mathcal{P})} \leq 1; \)

ii. \( Q(\hat{f})(\omega) = E_\omega(f(\omega)|\mathcal{P}^1) \) for any \( \hat{f} \in L_M(\mathcal{P}, \mathcal{P}) \) and for almost all \( \omega \in \Omega. \)

**Proof:**

Let \( Q_\omega \) be a linear contractions on \( L_1(\mathcal{P}_\omega, \mathcal{P}_\omega) \) constructed in Theorem 3.1, such that

\( Q(\hat{f})(\omega) = Q_\omega(f(\omega)) \) for \( \hat{f} \in L_1(\mathcal{P}, \mathcal{P}) \) and for almost all \( \omega \in \Omega. \)

Since \( \| Q_\omega \|_{L_1(\mathcal{P}_\omega, \mathcal{P}_\omega)} \leq 1 \) and \( Q_\omega(1) = 1 \) by (Krasnoselskii et al., 1961) (II. sec. 4. Item 6) we have that \( \| Q_\omega \|_{L_M(\mathcal{P}_\omega, \mathcal{P}_\omega)} \leq 1 \).

Using Proposition 2.3 (Zakirov & Chilin, 2009) we get that

\( \| Q(\hat{f}) \|_{L_M(\mathcal{P}, \mathcal{P})} = \| Q_\omega(f(\omega)) \|_{L_M(\mathcal{P}_\omega, \mathcal{P}_\omega)} \leq \| f(\omega) \|_{L_M(\mathcal{P}_\omega, \mathcal{P}_\omega)} \)

for almost all \( \omega \in \Omega, \) i.e.

\[\| Q(\hat{f}) \|_{L_M(\mathcal{P}, \mathcal{P})} \leq \| \hat{f} \|_{L_p(\mathcal{P}, \mathcal{P})} \quad \text{or} \quad \| Q \|_{L_M(\mathcal{P}, \mathcal{P})} \leq 1.\]  \hspace{1cm} (15)
As $Q_{\omega}^2 = Q_{\omega} \parallel Q_{\omega} \parallel_{L_M(\mathcal{V}_{\omega}\mu_{\omega})} - L_M(\mathcal{V}_{\omega}\mu_{\omega}) \leq 1$, by (Rao, 1965) there exists a unique regular sub-algebra $\mathcal{V}_{\omega}^1$ of $\mathcal{V}_{\omega}$, such that

$$Q_{\omega} = E_{\omega}(\cdot \mid \mathcal{V}_{\omega}^1).$$

Hence $Q(f)(\omega) = E_{\omega}(f(\omega) \mid \mathcal{V}_{\omega}^1)$ for any $f \in L_M(\mathcal{V}, \mu)$ and for almost all $\omega \in \Omega$.

**Theorem 3.3.** Let $\hat{f} \in L_1(\mathcal{V}, \mu)$ then

- $|E(\hat{f} | \hat{\Theta}^1)| \leq E(|\hat{f}| \mid \hat{\Theta}^1)$;
- Let $\hat{f}_n \in L_1(\mathcal{V}, \mu)$ such that
- 1) $|\hat{f}_n| \leq \hat{g} \in L_1(\mathcal{V}, \mu)$ and
- 2) $\hat{f}_n \overset{(o)}{\rightarrow} \hat{f}$ then

$$E(\hat{f}_n \mid \Theta^1) \overset{(o)}{\rightarrow} E(\hat{f} \mid \Theta^1).$$  (17)

**Proof:**

(i) The numerical case proof is applicable here (see (Doob, 1953)).

(ii) From 1) it follows that $|f_n(\omega)| \leq g(\omega) \in L_1(\mathcal{V}_{\omega}, \mu_{\omega})$ then by (Doob, 1953) $E_{\omega}(f_n(\omega) \mid \mathcal{V}_{\omega}^1) \overset{(o)}{\rightarrow} E_{\omega}(f(\omega) \mid \mathcal{V}_{\omega}^1)$ for almost all $\omega \in \Omega$.

Since

$$E(\hat{f}_n \mid \hat{\Theta}^1) - E(\hat{f} \mid \hat{\Theta}^1) = E_{\omega}(f_n(\omega) \mid \mathcal{V}_{\omega}^1) - E_{\omega}(f(\omega) \mid \mathcal{V}_{\omega}^1)$$

(18)

we get that $E(\hat{f}_n \mid \Theta^1) - E(f \mid \Theta^1) \overset{(o)}{\rightarrow} 0$ for almost all $\omega \in \Omega$. Then using Theorem 4.1 (Ganiev, 2006) we obtain

$$E(\hat{f}_n \mid \Theta^1) \overset{(o)}{\rightarrow} E(\hat{f} \mid \Theta^1).$$  (19)
Theorem 3.4. If $T: L_0 \to L_0$ is $L_0$-linear and $L_0$-bounded operator and $f \in L_1(\mathcal{P}, \mathcal{M})$, then

$$E(T(f)|\mathcal{M}) = T(E(f)|\mathcal{M}).$$

4 Conclusion

Any positive projective contractions $Q$ with $Q1 = 1$ in the Orlicz — Kantorovich lattices $L_M(\mathcal{P}, \mathcal{M})$ can be represented in the form

$$Q(f)(\omega) = E_{\omega}(f(\omega)|\mathcal{M})$$

for any $f \in L_M(\mathcal{P}, \mathcal{M})$ and for almost all $\omega \in \Omega$, where $E_{\omega}(.|\mathcal{M})$ is conditional expectation operator.

Acknowledgements.
The first author acknowledges the MOHE Grant FRGS13-071-0312.

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